Bound-state properties from field-theory correlators

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Abstract. We discuss the details of calculating hadron properties from the OPE for correlators of quark currents in QCD, which constitutes the basis of the method of QCD sum rules. The main emphasis is laid on gaining control over the systematic uncertainties of the hadron parameters obtained within this method. We start with examples from quantum mechanics, where bound-state properties may be calculated independently in two ways: exactly, by solving the Schrödinger equation, and approximately, by the method of sum rules. Knowing the exact solution allows us to control each step of the sum-rule extraction procedure. On the basis of this analysis, we formulate several improvements of the method of sum rules. We then apply these modifications to the analysis of the decay constants of heavy mesons.

1. Introduction

The method of dispersive sum rules [1] is one of the widely used methods for obtaining properties of ground-state hadrons in QCD. The method involves two steps: (i) one calculates the relevant correlator in QCD at relatively small values of the Eucledian time; (ii) one applies numerical procedures suggested by quark-hadron duality in order to isolate the ground-state contribution from this correlator. These numerical procedures cannot determine a single value of the ground-state parameter but should provide the band of values containing the true hadron parameter. This band is a systematic, or intrinsic, uncertainty of the method of sum rules.

An unbiased judgement of the reliability of the extraction procedures adopted in the method of sum rules may be acquired by applying these procedures to problems where the ground-state parameters may be found independently and exactly as soon as the parameters of theory are fixed. Presently, only quantum-mechanical potential models provide such a possibility: (i) the bound-state parameters (masses, wave functions, form factors) are known precisely from the Schrödinger equation; (ii) direct analogues of the QCD correlators may be calculated exactly.

Making use of these models, we studied the extraction of ground-state parameters from different types of correlators: namely, the ground-state decay constant from two-point vacuum-to-vacuum correlator [2], the form factor from three-point vacuum-to-vacuum correlator [3], and the form factor from vacuum-to-hadron correlator [4]. We have demonstrated that the standard procedures adopted in the method of sum rules not always work properly: the true value of the bound-state parameter was shown to lie outside the band obtained according to the standard criteria. These results gave us a solid ground to claim that also in QCD the actual accuracy of the method may be worse than the accuracy expected on the basis of applying the standard criteria.

We realized that the main origin of these problems of the method originate from an over-simplified model for hadron continuum which is described as a perturbative contribution above a constant Borel-parameter independent effective continuum threshold. We introduced the notion of the *exact* effective continuum threshold, which corresponds to the true bound-state parameters: in potential models the true hadron parameters—decay constant and form factor—are known and the exact effective continuum thresholds for different correlators may be calculated. We have demonstrated that the exact effective continuum threshold (i) is not a universal quantity and depends on the correlator considered (i.e., it is in general different for two-point and three-point vacuum-to-vacuum correlators), (ii) depends on the Borel parameter and, for the form-factor case, also on the momentum transfer [3, 4, 5].

In recent publications [6] we proposed a new algorithm for extracting the parameters of the ground state. The idea formulated in these papers is to relax the standard assumption of a Borel-parameter independent effective continuum threshold and to allow for a Borel-parameter dependent quantity. This lecture explains the details of this procedure and its application to decay constants of heavy mesons.

2. OPE and sum rule in quantum-mechanical potential model

Let us start with quantum-mechanical potential model described by Hamiltonian

$$H = H_0 + V(r); \qquad H_0 = \frac{k^2}{2m}, \qquad V(r) = V_{\rm conf}(r) - \frac{\alpha}{r}. \tag{1}$$

Polarization operator $\Pi(E)$ and its Borel transform $\Pi(T)$ [$E \to T$, $1/(H-E) \to \exp(-HT)$, T the Borel parameter] are defined via the full Green function $G(E) = 1/(H-E)$ [7]:

$$\Pi(E) = \langle \vec{r}_f = 0 | \frac{1}{H-E} | \vec{r}_i = 0 \rangle, \qquad \Pi(T) = \langle \vec{r}_f = 0 | \exp(-HT) | \vec{r}_i = 0 \rangle. \tag{2}$$

The expansion of $G(E)$ and $\Pi(E)$ in powers of the interaction is obtained with the help of the Lippmann-Schwinger equation

$$G(E) = G_0(E) - G_0(E) V G_0(E) + G_0(E) V G_0(E) V G_0(E) + \dots, \tag{3}$$

where $G_0(E) = 1/(H_0 - E)$. Since the interaction contains now two parts, $V_{\rm conf}(r)$ and $\frac{\alpha}{r}$, the expansion (3) is a double expansion in powers of $V_{\rm conf}$ and α. E.g., for the case $V_{\rm conf}(r) = \frac{m\omega^2 r^2}{2}$ one easily obtains the corresponding double expansion in powers of α and ωT:

$$\begin{aligned}
\Pi_{\rm OPE}(T) &= \Pi_{\rm pert}(T) + \Pi_{\rm power}(T), \quad \Pi_{\rm pert}(T) = \left(\frac{m}{2\pi T}\right)^{3/2} \left[1 + \sqrt{2\pi m T}\alpha + \frac{1}{3} m\pi^2 T \alpha^2\right], \\
\Pi_{\rm power}(T) &= \left(\frac{m}{2\pi T}\right)^{3/2} \left[-\frac{1}{4}\omega^2 T^2 \left(1 + \frac{11}{12}\sqrt{2\pi m T}\alpha\right) + \frac{19}{480}\omega^4 T^4\right].
\end{aligned} \tag{4}$$

One can see here a "perturbative contribution" (i.e. not containing the confining potential), and "power corrections" given in terms of the confining potential (including also mixed terms containing contributions from both Coulomb and confining potentials). A perturbative contribution may be written in the form of spectral representation [8] yielding

$$\Pi_{\rm OPE}(T) = \int\limits_0^\infty dz e^{-zT} \rho_{\rm pert}(z) + \Pi_{\rm power}(T), \ \rho_{\rm pert} = \left(\frac{m}{2\pi}\right)^{3/2} \left[2\sqrt{\frac{z}{\pi}} + \sqrt{2\pi m}\alpha + \frac{\pi^{3/2} m \alpha^2}{3\sqrt{z}}\right] \tag{5}$$

The "physical" representation for $\Pi(T)$—in the basis of hadron eigenstates—reads:

$$\Pi_{\rm phys}(T) = \langle \vec{r}_f = 0|\exp(-HT)|\vec{r}_i = 0\rangle = \sum_{n=0}^{\infty} R_n \exp(-E_n T), \quad R_n = |\Psi_n(\vec{r}=0)|^2. \tag{6}$$

Sum rule is the expression of the fact that the correlator may be calculated in two ways—using the basis of quark states (OPE) or confined bound states—leading to the same result:

$$\Pi_{\rm OPE}(T) = \Pi_{\rm phys}(T). \tag{7}$$

In order to isolate the ground-state contribution one needs the information about the excited states. A standard Ansatz for the hadron spectral density has the form [1]

$$\rho_{\rm phys}(z) = R_g \delta(z - E_g) + \theta(z - z_{\rm eff})\rho_{\rm pert}(z). \tag{8}$$

It assumes that the contribution of the excited states may be described by contributions of diagrams of perturbation theory above some effective continuum threshold $z_{\rm eff}$. This effective continuum threhsold (different from the physical continuum threhsold which is determined by hadron masses) is an additional parameter of the method of sum rules. Using Eq. (8) yields

$$R_g e^{-E_g T} = \int_0^{z_{\rm eff}} dz e^{-zT}\rho_{\rm pert}(z) + \Pi_{\rm power}(T) \equiv \Pi_{\rm dual}(T, z_{\rm eff}). \tag{9}$$

As soon as one knows $z_{\rm eff}$, one immediately obtains estimates for R_g and E_g, $R_{\rm dual}(T, z_{\rm eff})$ and $E_{\rm dual}(T, z_{\rm eff}) = -d_T \log \Pi(T, z_{\rm eff})$. These however depend on unphysical parameters T and $z_{\rm eff}$.

Eq. (8) is motivated by quark-hadron duality which claims that far above the threshold the hadron spectral density is well described by diagrams of perturbation theory. However, near the physical threshold—and this very region turns out to be essentail for the calculation of ground-state properties—the duality relation is violated.

The advantage of quantum mechanics is that the exact E_g and R_g may be obtained by solving Schrödinger equation [9]. We can then calculate $z_{\rm eff}$ from

$$R_g e^{-E_g T} = \int_0^{z_{\rm eff}} dz e^{-zT}\rho_{\rm pert}(z) + \Pi_{\rm power}(T). \tag{10}$$

The obtained "exact threshold" $z_{\rm eff}(T)$ is a slightly rising function of T [2].

3. OPE and sum rule in QCD

In QCD, the sum rule for the decay constants of heavy mesons has a similar form [10]

$$f_Q^2 M_Q^4 e^{-M_Q^2\tau} = \int_{(m_Q+m_u)^2}^{s_{\rm eff}} e^{-s\tau}\rho_{\rm pert}(s, \alpha, m_Q, \mu)\, ds + \Pi_{\rm power}(\tau, m_Q, \mu) \equiv \Pi_{\rm dual}(\tau, \mu, s_{\rm eff}) \tag{11}$$

In order to extract the decay constant one should fix the effective continuum threshold $s_{\rm eff}$ which should be a function of τ; otherwise the τ-dependences of the l.h.s. and the r.h.s. of (11) do not match each other. The exact $s_{\rm eff}$ corresponding to the exact hadron decay constant and mass on the l.h.s. is of course not known. The extraction of hadron parameters from the sum rule consists therefore in attempting (i) to find a good approximation to the exact threshold and (ii) to control the accuracy of this approximation. For further use, we define the dual decay constant $f_{\rm dual}$ and the dual invariant mass $M_{\rm dual}$ by relations

$$f_{\rm dual}^2(\tau) = M_Q^{-4} e^{M_Q^2\tau}\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)), \quad M_{\rm dual}^2(\tau) = -\frac{d}{d\tau}\log \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)). \tag{12}$$

4. Extraction of the decay constant: QCD vs. potential model [11]

If the mass of the ground state is known, the deviation of the dual mass from the actual mass of the ground state gives an indication of the contamination of the dual correlator by excited states. Assuming a specific functional form of the effective threshold and requiring the least deviation of the dual mass (12) from the known ground-state mass in the τ-window leads to a variational solution for the effective threshold. As soon as the latter has been fixed, one calculates the decay constant from (12). **Our algorithm for extracting ground-state parameters reads:**

(i) Consider a set of Polynomial τ-dependent Ansaetze for $s_{\rm eff}$: $s_{\rm eff}^{(n)}(\tau) = s_0^{(n)} + s_1^{(n)}\tau + s_2^{(n)}\tau^2 + \dots$.

(ii) Calculate the dual mass for these τ-dependent thresholds and minimize the squared difference between the "dual" mass $M_{\rm dual}^2$ and the known value M_B^2 in the τ-window. This gives the parameters of the effective continuum thresholds $s_i^{(n)}$.

(iii) Making use of the obtained thresholds, calculate the decay constant.

(iv) Take the band of values provided by the results corresponding to **linear**, **quadratic**, and **cubic** effective thresholds as the characteristic of the intrinsic uncertainty of the extraction procedure.

Figure 1 shows the application of our algorithm in quantum mechanics and in QCD. In both cases a very similar pattern emerges; the reproduction of the dual mass considerably improves for the τ-dependent quantities indicating that the dual correlator with a τ-dependent threshold isolates the contribution of the ground state much better than the dual correlator with a standard τ-independent threshold. As a consequence, the accuracy of the extracted hadron observable improves considerably.

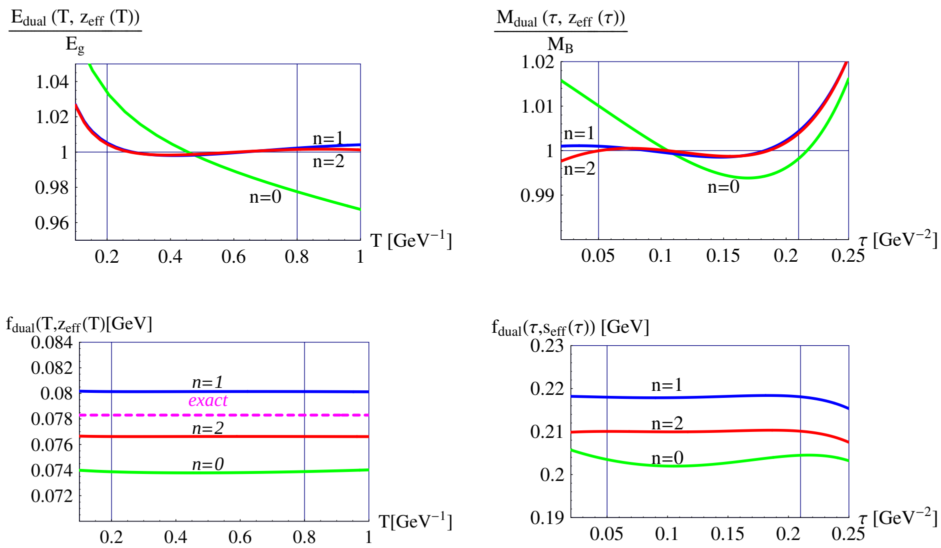

Figure 1. The outcome of our algorithm in potential model (left) and in QCD (right).

5. Decay constants of D and D_s mesons [12]

The application of our extraction procedures leads to the following results:

$$f_D = 206.2 \pm 7.3_{\rm (OPE)} \pm 5.1_{\rm (syst)}\,{\rm MeV}, \qquad f_{D_s} = 245.3 \pm 15.7_{\rm (OPE)} \pm 4.5_{\rm (syst)}\,{\rm MeV}. \tag{13}$$

The τ-dependent threshold is a crucial ingredient for a successful extraction of the decay constant from the sum rule (Fig. 2). Obviously, the standard τ-independent approximation leads to much lower value for f_D which lies rather far from the data and the lattice results.

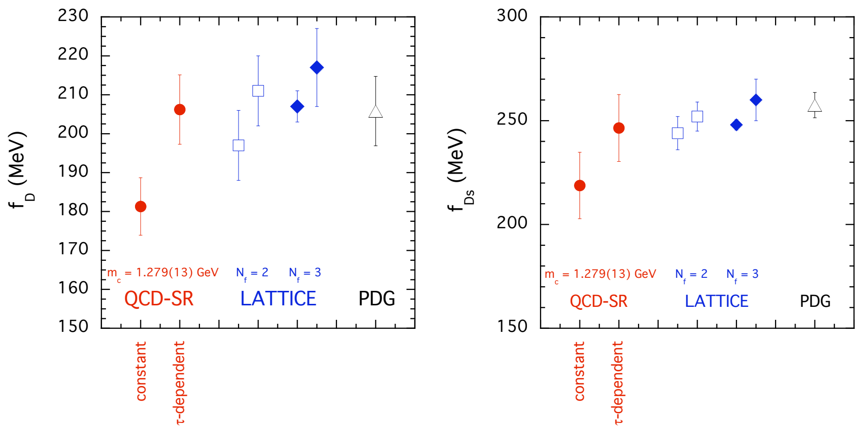

Figure 2. Our results for f_D and f_{Ds} [12].

6. Conclusions

The effective continuum threshold $s_{\rm eff}$ is an important ingredient of the method of dispersive sum rules which determines to a large extent the numerical values of the extracted hadron parameter. Finding a criterion for fixing $s_{\rm eff}$ poses a problem in the method of sum rules.

• $s_{\rm eff}$ depends on the *external kinematical variables* (e.g., momentum transfer in sum rules for 3-point correlators and light-cone sum rules) and *''unphysical'' parameters* (renormalization scale μ, Borel parameter τ). Borel-parameter τ-dependence of $s_{\rm eff}$ emerges naturally when trying to make quark-hadron duality more accurate.

• We proposed a new algorithm for fixing τ-dependent $s_{\rm eff}$, for those problems where the bound-state mass is known. We have tested that our algorithm leads to more accurate values of ground-state parameters than the "standard" algorithms used in the context of dispersive sum rules before. Moreover, our algorithm allows one to probe *"intrinsic"* uncertainties related to the limited accuracy of the extraction procedure in the method of QCD sum rules.

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